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A perturbed Whittaker–Kotel'nikov–Shannon sampling theorem

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ABSTRACT

The sampling theorem of Whittaker (1915) [31], Kotel'nikov (1933) [25] and Shannon (1949) [28] gives cardinal series representations for finite L^2-Fourier transforms at equidistant sampling points. Here we investigate the situation when the Fourier transform is replaced by a perturbed one. Thus the kernel of the transform will be of the form $\exp(-ixt) + \varepsilon(x,t)$, instead of $\exp(-ixt)$ in the unperturbed case. The perturbed kernel arises from first order eigenvalue problems with rank one perturbations.

1. Introduction

Let us consider the Paley–Wiener space $PW^2_\pi(\mathbb{R})$ of all $L^2(\mathbb{R})$-functions whose Fourier transforms vanish outside $[-\pi,\pi]$. Thus $f \in PW^2_\pi(\mathbb{R})$ if there exists a unique $L^2(-\pi,\pi)$-function $g(\cdot)$ such that

$$f(t) = \int_{-\pi}^{\pi} g(x)e^{-ixt}\,dx. \tag{1.1}$$

The space $PW^2_\pi(\mathbb{R})$ is a Hilbert space which has been characterized via the following theorem, which is due to Paley and Wiener [26, p. 13]; [27, p. 243]. See also [7]. In Engineering terminology elements of the Paley–Wiener space $PW^2_\pi(\mathbb{R})$ are called band-limited signals with band-width π.

Theorem I. *The space $PW^2_\pi(\mathbb{R})$ coincides with the class of all $L^2(\mathbb{R})$-entire functions with exponential type π.*

The classical sampling theorem of Whittaker–Kotel'nikov–Shannon (WKS), for which we will derive its perturbed counterpart states:

Theorem II. *Let $f \in PW^2_\pi(\mathbb{R})$. Then f can be recovered from its values at the integers via the sampling series*

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}, \quad t \in \mathbb{C}. \tag{1.2}$$

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The convergence of series (1.2) *is uniform on* $\mathbb{R}$ *and on compact subsets of* $\mathbb{C}$ *and it is absolute on* $\mathbb{C}$. *Moreover the series* (1.2) *converges in the* $L^2(\mathbb{R})$*-norm.*

See [10,25,28,31]. The proof of (1.2) could be derived in several ways, see e.g. [20]. The WKS sampling theorem has many applications in signal processing, see e.g. [23].

In the works of Campbell [11] and Everitt and Poulkou [18], the authors indicated that Theorem II could be established in connection with the first order eigenvalue problem

$$\ell(y) := iy' = ty, \quad -\pi \leqslant x \leqslant \pi,\ t \in \mathbb{C}, \tag{1.3}$$

$$V(y) := y(\pi) - y(-\pi) = 0. \tag{1.4}$$

In this setting, the kernel $\exp(-ixt)$ of (1.1) is a solution of (1.3). The sampling points in (1.2), i.e. the integers, are exactly the eigenvalues of problem (1.3)–(1.4). It is worthwhile to notice that the set of eigenfunctions of (1.3)–(1.4), namely $\{e^{-inx}\}_{n=-\infty}^{\infty}$ is generated by a single function, namely the solution (kernel). In another direction, still related to (1.3)–(1.4), Haddad et al. [19] derived Theorem II using Green's function of (1.3)–(1.4). These ideas and results were extended in several directions, see e.g. [1,3,6,9,15–17].

Our main purpose of this article is to investigate the possibilities to derive sampling expansions for perturbed Fourier transforms of the type

$$f(t) = \int_{-\pi}^{\pi} g(x)\big[e^{-ixt} + \varepsilon(x,t)\big]\,dx, \quad g(\cdot) \in L^2(-\pi,\pi), \tag{1.5}$$

where $\varepsilon(x,t)$ is a given function. There are many applications of perturbed Fourier transforms with good practical results. This applications involve, for the so-called canonical transforms, pattern recognition; radar system analysis; filter design and optics, see e.g. [21,22,30,32,34,33]. One of the definitions of the canonical Fourier transform is the perturbed transform

$$\widehat{f}(t) = \mathcal{L}_M[f](t) = \begin{cases} \sqrt{\frac{1}{2i\pi b}}\int_{-\infty}^{\infty} f(x)\exp[\frac{i}{2}(\frac{a}{b}x^2 - \frac{2}{b}xt + \frac{d}{b}t^2)]\,dx, & b \neq 0,\\ \sqrt{d}\exp[\frac{i}{2}c\,dt^2]f(dt), & b = 0, \end{cases} \tag{1.6}$$

where a, b, c, d are real numbers satisfying

$$\det(M) = \det\begin{pmatrix} a & b\\ c & d\end{pmatrix} = 1. \tag{1.7}$$

The case $b = 0$ is trivial and of no interest, cf. [22]. When $b = 1$ the canonical transform is the perturbed one, (1.5), with

$$\varepsilon(x,t) = e^{-ixt}\left[\sqrt{\frac{1}{2i\pi b}}\exp\left[\left(\frac{ia}{2}x^2 + \frac{id}{2}t^2\right)\right] - 1\right]. \tag{1.8}$$

Notice that the plus or minus signs in the exponential could be interchanged because they appear in the transform and its inverse alternatively. Sampling theorems for band-limited canonical type transforms are a major task of many signal processing and optical applications, see e.g. [21,22,30,33]. See also [24] for applications of generalized Fourier transforms in optics, and in particular in diffraction, i.e. the phenomena resulting from the passage of monochromatic light through a sufficiently small aperture, which causes the deviation of light from its original direction of propagation.

In the current paper, we derive sampling theorems for a generalized class of perturbed Fourier transform. We do not study the problem from its very general point of view, but we investigate the case when the kernel of (1.5) arises from an eigenvalue problem of the type (1.3)–(1.4) when it has an additional rank-one perturbation operator. In this case the resulting kernel will be of the type mentioned above. One of the problems in this situation is that unlike problem (1.3)–(1.4), we cannot always find a kernel which generates all eigenfunctions of the problem since the eigenvalues are not necessarily simple, but a finite number of them may be double. We give a brief account on the spectral analysis of the perturbed problem. The perturbed sampling theorems are contained in Section 3 below. They are of Lagrange-type interpolations. Examples, comparisons are given in the last section. A treatment for sampling theory associated with second order perturbed Sturm–Liouville problems could be found in [2] and a discrete counterpart is established in [5].

2. A spectral analysis

Consider the boundary-value problem which consists of the integro-differential equation

$$\ell_r(y) := iy' + r(x)\int_{-\pi}^{\pi} r(\tau)y(\tau)\,d\tau = ty, \quad -\pi \leqslant x \leqslant \pi, \tag{2.1}$$

and condition (1.4). Here $r(\cdot)$ is a continuous real-valued function on $[-\pi,\pi]$. This problem, (2.1), (1.4), differs from problem (1.3)–(1.4) only by the existence of the rank-one perturbation of (2.1). It will be denoted by Π_r to be distinguished from (1.3)–(1.4), which we denote by Π. We can see that problem Π_r is self adjoint with real eigenvalues only. Nevertheless, problem Π_r has some properties that will change the sampling results. For instance we cannot always find a function that generates all eigenfunctions like $\exp(-ixt)$ in (1.1) above. Since as we see below, the eigenvalues are not necessarily simple, we may need to use Green's function as in [6] or the method of [4,16,17], where the eigenfunctions are generated by more than one function. This is why we need to investigate some of the spectral properties of Π_r.

Let us first find the general solution of Eq. (2.1). We use the technique established by Catchpole in [12] and Stakgold in [29, pp. 405–409]. Let us denote the constant $\int_{-\pi}^{\pi} r(\tau)y(\tau)\,d\tau$ by ρ. Then Eq. (2.1) is nothing but

$$y' + ity = i\rho r(x), \tag{2.2}$$

which has the solution

$$y = e^{-itx}\left(i\rho\int_{-\pi}^{x} r(\tau)e^{it\tau}\,d\tau + c\right) = ce^{-itx} + \rho P(x,t), \tag{2.3}$$

where c is an arbitrary constant, and

$$P(x,t) := ie^{-itx}\int_{-\pi}^{x} r(\tau)e^{it\tau}\,d\tau. \tag{2.4}$$

Note that $P(\cdot,t)$ is the unique solution of the nonhomogeneous initial value problem

$$y' + ity = ir(x), \qquad y(-\pi) = 0. \tag{2.5}$$

Therefore, a solution ϕ of Eq. (2.1) satisfies the nonhomogeneous Fredholm integral equation of the second kind

$$\phi = ce^{-itx} + \langle \phi, r\rangle P, \qquad \langle u, v\rangle := \int_{-\pi}^{\pi} u(\tau)\bar{v}(\tau)\,d\tau. \tag{2.6}$$

Multiplying (2.6) by $r(\cdot)$ and integrating over $[-\pi,\pi]$ yield

$$\langle \phi, r\rangle = c\int_{-\pi}^{\pi} e^{-itx} r(x)\,dx + \langle \phi, r\rangle\langle P, r\rangle. \tag{2.7}$$

Thus we get the following equation involving ρ

$$\rho C(t) = c\int_{-\pi}^{\pi} e^{-itx} r(x)\,dx, \tag{2.8}$$

where

$$C(t) := 1 - \langle P, r\rangle = 1 - i\int_{-\pi}^{\pi}\int_{-\pi}^{x} e^{-it(x-\tau)} r(\tau)r(x)\,d\tau\,dx. \tag{2.9}$$

Since the determination of ρ requires a division on $C(t)$ in (2.8), then the derivation of the general solution of (2.1) depends on whether $C(t) = 0$ or not.

Lemma 2.1. *If $C(t) \neq 0$ for some $t \in \mathbb{C}$, then the general solution of* (2.1) *has the form*

$$\phi_c(x,t) := c\left(\varphi + \frac{\langle \varphi, r\rangle}{C(t)} P\right), \qquad \varphi(x,t) := e^{-itx}, \tag{2.10}$$

where c is an arbitrary constant. When $C(t) = 0$ for some $t \in \mathbb{C}$, then the general solution of (2.1) *has the form*

$$\psi(x,t) := c\varphi(x,t) + \gamma P(x,t), \tag{2.11}$$

where c; γ are arbitrary constants.

Proof. If $C(t) \neq 0$, then solving (2.8) and substituting in (2.3), we get (2.10). Now, let $C(t) = 0$, for some $t \in \mathbb{C}$. We first see that $\varphi(\cdot, t)$ is a solution. For, we can see from (2.6) that $\langle \varphi, r \rangle = 0$. Multiplying both sides of (2.6) by $r(x)$ and integrating over $[-\pi, \pi]$, we obtain

$$\langle \phi(\cdot, t), r(\cdot) \rangle = c \langle \varphi(\cdot, t), r(\cdot) \rangle + \langle \phi(\cdot), r(\cdot) \rangle \langle P(\cdot, t), r(\cdot) \rangle.$$

But $\langle P(\cdot, t), r(\cdot) \rangle = 1$, since $C(t) = 0$. Hence

$$\langle \phi(\cdot, t), r(\cdot) \rangle = c \langle \varphi(\cdot, t), r(\cdot) \rangle + \langle \phi(\cdot), r(\cdot) \rangle,$$

implying that $\langle \varphi, r \rangle = 0$. Now direct substitutions in (2.1) proves that $\varphi(\cdot, t)$ is one of its solutions. Also direct computations prove that $P(\cdot, t)$ is another linearly independent solution. Hence, when $C(t) = 0$ for some t, any solution of (2.1) has the form (2.11). □

The previous lemma shows that, when $C(t)$ does not vanish, the linear space of solutions of Eq. (2.1) is a one-dimensional space as is the case of (1.3). Also a solution is uniquely determined by one initial condition. So in this case all eigenvalues when $C(t) \neq 0$ are simple. But when $C(t)$ vanishes at a certain point, then the linear space of solutions of Eq. (2.1) is a two-dimensional space and an initial condition determines infinitely many solutions. This is a serious change that will affect the spectral analysis as well as the sampling results.

In the following we investigate the multiplicity of the eigenvalues of Π_r. Unlike unperturbed problems, one expects that the eigenvalues may be double. As is seen above we will have two cases, i.e. when $C(t) = 0$ and when $C(t) \neq 0$. From now on $\phi(\cdot, t)$ denotes the function

$$\phi(x, t) := \phi_1(x, t) := \varphi + \frac{\langle \varphi, r \rangle}{C(t)} P, \quad C(t) \neq 0. \tag{2.12}$$

If $C(t) \neq 0$, then t is an eigenvalue if and only if

$$\Delta(t) := V\big(\phi(\cdot, t)\big) = 0, \tag{2.13}$$

recall (1.4) for definitions. When $C(t) = 0$, then we have the following lemma that indicates that this value of t must be an eigenvalue and hence it is real.

Lemma 2.2. *Let $C(t) = 0$ for some $t \in \mathbb{C}$. Then t is a simple eigenvalue of Π_r with an eigenfunction $P(\cdot, t)$ if it is not an eigenvalue of Π. Otherwise it is a double eigenvalue. Consequently all zeros of $C(t)$, if any, are real.*

Proof. Let $C(t) = 0$ for some $t \in \mathbb{C}$. In this case we can see that $P(\cdot, t)$ is an eigenfunction of Π_r according to this t. Indeed, since

$$\langle \varphi, r \rangle = \int_{-\pi}^{\pi} e^{-it\tau} r(\tau)\, d\tau = 0,$$

and $r(\cdot)$ is real-valued, then

$$\int_{-\pi}^{\pi} e^{it\tau} r(\tau)\, d\tau = 0.$$

Hence, $P(\pi, t) = 0$. Moreover, from (2.5), $P(-\pi, t) = 0$. Therefore, $V(P(\cdot, t)) = 0$, i.e. $P(\cdot, t)$ satisfies the boundary condition (1.4) in addition to Eq. (2.1). Hence $P(\cdot, t)$ is an eigenfunction of Π_r. This proves that t must be real, since from the self-adjointness of problem (2.1)–(1.4), all eigenvalues are real. The eigenvalue t will have another eigenfunction $\psi = \varphi + \gamma P$ if

$$0 = V(\psi) = V(\varphi) + \gamma V(P) = V(\varphi), \tag{2.14}$$

i.e. t is an eigenvalue of Π. The proof is accomplished by noting that all eigenvalues of Π are simple and that $P(\cdot, t)$ cannot be an eigenfunction of Π. □

As we saw above, double eigenvalues, if any, are integers and zeros of $C(t)$. Fortunately, as is seen in the next corollary, the number of double eigenvalues is finite.

Corollary 2.3. *The function $C(t)$ has at most a finite number of real zeros.*

Proof. The function $C(t)$ cannot have real zeros with finite limit points since otherwise, the entire function $C(t)$ vanishes identically. This will lead to the contradiction that problem (2.1), (1.4) has non-real eigenvalues. Then the only possible limit points for the zeros of $C(t)$ are $\pm\infty$. Assume that $t \in \mathbb{R}$. Since

$$C(t) = 1 - i \int_{-\pi}^{\pi} e^{-itx} r(x) \int_{-\pi}^{x} r(\tau) e^{it\tau}\, d\tau\, dx,$$

then by Riemann–Lebesgue Lemma [8, p. 167], we have for $x \in [-\pi, \pi]$,

$$\lim_{t\to\pm\infty} \int_{-\pi}^{x} r(\tau) e^{it\tau}\, d\tau = 0. \tag{2.15}$$

For $x \in [-\pi, \pi]$, $e^{-itx} r(x)$ is bounded on $\mathbb{R}$ as a function of t and therefore

$$\lim_{t\to\pm\infty} f(x,t) := \lim_{t\to\pm\infty} e^{-itx} r(x) \int_{-\pi}^{x} r(\tau) e^{it\tau}\, d\tau = 0. \tag{2.16}$$

Hence, for any sequence t_n of real numbers with $\lim_{n\to\infty} t_n = \pm\infty$, we obtain

$$\lim_{n\to\infty} f(x, t_n) = 0, \quad x \in [-\pi, \pi].$$

Also

$$\left| f(x,t) \right| \leqslant \left| r(x) \right| \int_{-\pi}^{x} \left| r(\tau) \right| d\tau := g(x), \quad t \in \mathbb{R}.$$

The function $g(x) \in L^1(-\pi, \pi)$. Hence, from Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x, t_n)\, dx = 0,$$

for all real sequences t_n with $t_n \to \pm\infty$. Thus

$$\lim_{t\to\pm\infty} \int_{-\pi}^{\pi} f(x,t)\, dx = 0.$$

Then $C(t)$, cannot have large zeros since $\lim_{t\to\pm\infty} C(t) = 1$. Therefore, $C(t)$ cannot have more than a finite number of real zeros. □

Herewith an example of a problem where $C(t)$ never vanishes on $\mathbb{C}$. However, we could see that the forms are complicated.

Example 2.4. Let us consider the eigenproblem

$$iy' + x \int_{-\pi}^{\pi} \tau y(\tau)\, d\tau = ty, \qquad y(\pi) - y(-\pi) = 0. \tag{2.17}$$

Simple calculations lead to

$$P(x,t) = \frac{ie^{-2itx}(e^{it(x+\pi)}(i\pi t - 1) + itx + 1)}{t^2}, \qquad C(t) = \frac{2i(\sin(\pi t) - \pi t\cos(\pi t))^2}{t^4} + 1. \tag{2.18}$$

It is readily seen that $C(t) \neq 0$ for all $t \in \mathbb{R}$, simply because $\Re(C(t)) = 1 \neq 0$. Consequently, $C(t)$ never vanishes on $\mathbb{C}$. The fundamental solution becomes

$$\phi(x,t) = \frac{2ie^{-2itx}(e^{it(x+\pi)}(i\pi t - 1) + itx + 1)(\sin(\pi t) - \pi t\cos(\pi t))}{t^4(\frac{2i(\sin(\pi t) - \pi t\cos(\pi t))^2}{t^4} + 1)} + ie^{-itx}, \tag{2.19}$$

which can be set in the form

$$\phi(x,t) = ie^{-itx}\left[\frac{2(e^{-it\pi}(i\pi t-1)+ie^{-2itx}(itx+1))(\sin(\pi t)-\pi t\cos(\pi t))}{t^4(\frac{2i(\sin(\pi t)-\pi t\cos(\pi t))^2}{t^4}+1)}+1\right]. \tag{2.20}$$

The characteristic determinant $\Delta(t)$ is

$$\Delta(t) = \frac{2ie^{-2i\pi t}(i\pi t+e^{2i\pi t}(i\pi t-1)+1)(\sin(\pi t)-\pi t\cos(\pi t))}{t^4(\frac{2i(\sin(\pi t)-\pi t\cos(\pi t))^2}{t^4}+1)}+2i\sin\pi t. \tag{2.21}$$

Here all eigenvalues will be the zeros of $\Delta(t)$, which cannot be computed explicitly. In the last section we will give concrete examples, where all spectral quantities can be concretely computed.

The construction of Green's function of Π_r is important because on the one hand it is needed for the proof of the completeness of the eigenfunctions, and it plays a major role in the sampling theory when we have double eigenvalues on the other hand. We again use the technique of Stakgold [29, pp. 405–409] to construct the Green's function of Π_r. Let $G(x,\xi,t)$ be Green's function of the unperturbed problem Π. Then, for $t \in \mathbb{C}-\mathbb{Z}$, we have

$$\begin{aligned} G(x,\xi,t) &= \frac{i}{1-e^{2it\pi}}\begin{cases} e^{it(\xi-x+2\pi)}, & \xi \leqslant x, \\ e^{it(\xi-x)}, & x \leqslant \xi \end{cases} \\ &= \sum_{n\in\mathbb{Z}} \frac{e^{-inx}e^{in\xi}}{2\pi(n-t)}, \end{aligned} \tag{2.22}$$

where the convergence of the sum is uniform with respect to $x,\xi \in [-\pi,\pi]$ and is pointwise with respect to t and is in $L^2((-\pi,\pi)\times(-\pi,\pi))$, see e.g. [14, pp. 190–202]. If $f(\cdot)$ is continuous on $[-\pi,\pi]$, and t is not an eigenvalue of Π, then the inhomogeneous problem

$$iy' - ty = f, \qquad V(y) = 0, \tag{2.23}$$

has the unique solution

$$y(x) = \int_{-\pi}^{\pi} G(x,\xi,t) f(\xi)\, d\xi. \tag{2.24}$$

The following lemma is needed for the construction of Green's function.

Lemma 2.5. *If t is not an eigenvalue of Π_r, then*

$$1+\langle \mathfrak{G}_t r, r\rangle \neq 0, \qquad (\mathfrak{G}_t r)(x) := \int_{-\pi}^{\pi} G(x,\xi,t) r(\xi)\, d\xi. \tag{2.25}$$

Proof. Since $G(x,\xi,t)$ is the Green's function of problem (2.23), then the function

$$(\mathfrak{G}_t r)(x) = \int_{-\pi}^{\pi} G(x,\xi,t) r(\xi)\, d\xi = -P(x,t),$$

uniquely solves the problem

$$iy' - ty = r(x), \qquad V(y) = 0.$$

Now assume that (2.25) does not hold. Then $\langle \mathfrak{G}_t r, r\rangle = -1$. Hence $(\mathfrak{G}_t r)(x)$ is an eigenfunction of Π_r corresponding to the eigenvalue t, contradicting the assumption. □

In the following we seek a solution of the problem

$$iy' - ty + r\langle y, r\rangle = f, \qquad V(y) = 0. \tag{2.26}$$

Theorem 2.6. *Let t be not an eigenvalue of Π_r. Then problem* (2.26) *has the unique solution*

$$y(x) = \int_{-\pi}^{\pi} H(x,\xi,t) f(\xi)\,d\xi, \qquad H(x,\xi,t) := G(x,\xi,t) - \frac{(\mathfrak{G}_t r)(x)(\mathfrak{G}_t^* r)(\xi)}{1 + \langle \mathfrak{G}_t r, r\rangle}, \tag{2.27}$$

where $(\mathfrak{G}_t^* r)(\xi) := \overline{(\mathfrak{G}_{\bar t} r)}(\xi)$. *The function* $H(x,\xi,t)$ *is the Green's function of problem* Π_r *and it is uniquely defined.*

Proof. Let t be neither an eigenvalue of Π_r nor an eigenvalue of Π. Since $G(x,\xi,t)$ is the Green's function of (2.23), then the solution of (2.26) is given by

$$y(x) = (\mathfrak{G}_t f)(x) - \rho(\mathfrak{G}_t r)(x), \qquad \rho := \langle y, r\rangle. \tag{2.28}$$

Multiplying both sides of (2.28) by $r(x)$ and integrating over $[-\pi,\pi]$, yield

$$\rho\big[1 + \langle \mathfrak{G}_t r, r\rangle\big] = \langle \mathfrak{G}_t f, r\rangle. \tag{2.29}$$

Lemma 2.5 above, guarantees that

$$\rho = \frac{\langle \mathfrak{G}_t f, r\rangle}{1 + \langle \mathfrak{G}_t r, r\rangle} \tag{2.30}$$

is well defined. Hence

$$y(x) = (\mathfrak{G}_t f)(x) - \frac{\langle \mathfrak{G}_t f, r\rangle}{1 + \langle \mathfrak{G}_t r, r\rangle}(\mathfrak{G}_t r)(x). \tag{2.31}$$

Since $G(x,\xi,t) = \overline{G}(\xi,x,\bar t)$, then formula (2.27) of $H(x,\xi,t)$ is achieved via

$$\begin{aligned} y(x) &= \int_{-\pi}^{\pi} G(x,\xi,t) f(\xi)\,d\xi - \frac{(\int_{-\pi}^{\pi} G(x,\xi,t) r(\xi)\,d\xi)(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} G(x,\xi,t) f(\xi) r(x)\,d\xi\,dx)}{1 + \langle \mathfrak{G}_t r, r\rangle} \\ &= \int_{-\pi}^{\pi} G(x,\xi,t) f(\xi)\,d\xi - \frac{(\mathfrak{G}_t r)(x)(\int_{-\pi}^{\pi} (\mathfrak{G}_t^* r)(\xi) f(\xi)\,d\xi)}{1 + \langle \mathfrak{G}_t r, r\rangle} \\ &= \int_{-\pi}^{\pi} H(x,\xi,t) f(\xi)\,d\xi. \end{aligned} \tag{2.32}$$

The uniqueness of $H(x,\xi,t)$ arises from the uniqueness of $G(x,\xi,t)$ and that of the constant ρ of (2.30). It remains to prove that $H(x,\xi,t)$ is well defined when t is not an eigenvalue of Π_r but it is an eigenvalue of Π, i.e. $t \in \mathbb{Z}$. Indeed, let $m \in \mathbb{Z}$ be not an eigenvalue of Π_r. From the simplicity of the poles of $G(x,\xi,t)$, we can find a neighborhood of m, $\Omega_m \subset \mathbb{C}$, say for which $G(x,\xi,t)$ can be written as

$$G(x,\xi,t) = G_1(x,\xi,t) + \frac{e^{-imx}e^{im\xi}}{2\pi(m-t)}, \quad t \in \Omega_m,\ t \neq m,$$

where $G_1(x,\xi,t)$ is analytic in Ω_m. Simple calculations yield, $t \in \Omega_m$, $t \neq m$,

$$\begin{aligned} H(x,\xi,t) &= G_1(x,\xi,t) + \frac{e^{-imx}e^{im\xi}}{2\pi(m-t)} - \frac{G_{12}(x,\xi,t) + \frac{G_{21}(x,\xi,t)}{m-t} + |\alpha|^2 \frac{e^{-imx}e^{im\xi}}{2\pi(m-t)^2}}{1 + G_{11}(t) + \frac{|\alpha|^2}{m-t}} \\ &= G_1(x,\xi,t) + \frac{(1 + G_{11}(t))e^{-imx}e^{im\xi} - 2\pi(m-t)G_{12}(x,\xi,t) - 2\pi G_{21}(x,\xi,t)}{2\pi((1+G_{11}(t))(m-t) + |\alpha|^2)}, \end{aligned}$$

where

$$\alpha := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{im\xi} r(\xi)\,d\xi,$$

$$G_{12}(x,\xi,t) := \left(\int_{-\pi}^{\pi} G_1(x,\xi,t) r(\xi)\,d\xi\right) \times \left(\int_{-\pi}^{\pi} G_1(x,\xi,t) r(x)\,dx\right),$$

$$G_{21}(x,\xi,t) := \frac{\overline{\alpha} e^{im\xi}}{\sqrt{2\pi}} \int_{-\pi}^{\pi} G_1(x,\xi,t) r(\xi)\, d\xi + \frac{\alpha e^{-imx}}{\sqrt{2\pi}} \int_{-\pi}^{\pi} G_1(x,\xi,t) r(x)\, dx,$$

$$G_{11}(t) := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} G_1(x,\xi,t) r(x) r(\xi)\, dx\, d\xi.$$

Noting that $\alpha \neq 0$, then $H(x,\xi,t)$ can be defined at m as a limit to be

$$H(x,\xi,m) = G_1(x,\xi,m) + \frac{1 + G_{11}(m) e^{-imx} e^{im\xi} - 2\pi G_{21}(x,\xi,m)}{2\pi |\alpha|^2}. \qquad \square$$

Corollary 2.7. *Problem Π_r has infinitely many real eigenvalues with no finite limit points and the eigenfunctions corresponding to different eigenvalues are orthogonal. The set of eigenfunctions is an orthogonal basis of $L^2(-\pi,\pi)$.*

Proof. Assume first that zero is not an eigenvalue of Π_r. Set

$$H(x,\xi) := H(x,\xi,0).$$

Hence $H(x,\xi)$ is Green's function of Π_r, i.e. any solution of

$$iy' + r\langle y, r\rangle = f, \qquad V(y) = 0, \tag{2.33}$$

is

$$y(x) = \int_{-\pi}^{\pi} H(x,\xi) f(\xi)\, d\xi. \tag{2.34}$$

Replacing f in (2.33) and (2.34) by ty, leads to the equivalence between Π_r and the Fredholm integral equation

$$y(x) = t \int_{-\pi}^{\pi} H(x,\xi) y(\xi)\, d\xi. \tag{2.35}$$

Noting that $H(x,\xi)$ is symmetric, then the eigenfunctions of Π_r are an $L^2(-\pi,\pi)$-basis if the kernel $H(x,\xi)$ is closed. This can be easily proved using the method of Stakgold [29, pp. 373–374]. The orthogonality holds for eigenfunctions corresponding to different eigenvalues. As for those linearly independent ones which belong to the same eigenvalue, we use the Gram–Schmidt procedure.

If zero is an eigenvalue of Π_r, completeness of the eigenfunctions can be deduced by replacing the eigenvalue parameter t by $t - c$, where c is a constant which is different from all eigenvalues of Π_r. Hence the new problem has the same eigenfunctions but zero is not an eigenvalue, completing the proof. $\square$

Lemma 2.8. *For a fixed $t \in \mathbb{R}$, $H(x,\xi,t)$ has the eigenfunction expansion*

$$H(x,\xi,t) = \sum_{n\in\mathbb{Z}} \frac{\phi_n(x)\overline{\phi}_n(\xi)}{t_n - t}, \quad t \neq t_n, \tag{2.36}$$

where $\{\phi_n(\cdot)\}_{n\in\mathbb{Z}}$ is a complete orthonormal set of eigenfunctions of Π_r and the convergence of (2.36) is in the $L^2((-\pi,\pi)\times(-\pi,\pi))$-norm.

Proof. Since $H(x,\xi)$ is a symmetric $L^2((-\pi,\pi)\times(-\pi,\pi))$-kernel, then it has the $L^2((-\pi,\pi)\times(-\pi,\pi))$-convergent expansion,

$$H(x,\xi) = \sum_{n\in\mathbb{Z}} \frac{\phi_n(x)\overline{\phi}_n(\xi)}{t_n}. \tag{2.37}$$

For $t \in \mathbb{R}$, $t \neq t_n$, $\{t_n - t\}_{n\in\mathbb{Z}}$ are the eigenvalues of the symmetric kernel $H(x,\xi,t)$ with the eigenfunctions $\{\phi_n(\cdot)\}_{n\in\mathbb{Z}}$. Hence

$$H(x,\xi,t) = \sum_{n\in\mathbb{Z}} \frac{\phi_n(x)\overline{\phi}_n(\xi)}{t_n - t}, \quad t \neq t_n, \tag{2.38}$$

where the convergence is in $L^2((-\pi,\pi)\times(-\pi,\pi))$-norm. $\square$

Now we connect Green's function $H(x,\xi,t)$ with the resolvent kernel of $H(x,\xi)$. For definitions of the resolvent kernel see [13,29].

Corollary 2.9. *$H(x,\xi,t)$ is the resolvent kernel of $H(x,\xi)$.*

Proof. Since the resolvent kernel $R_H(x,\xi,t)$, $t\neq t_n$ has expansion (2.36) [13,29], then $H(x,\xi,t)\equiv R_H(x,\xi,t)$ on $[-\pi,\pi]$. □

As a consequence of the previous corollary, expansion (2.36) holds for all $t\in\mathbb{C}$, $t\neq t_n$.

3. A perturbed WKS sampling theorem

As in the spectral analysis of the previous section, the sampling analysis associated with Π_r will be divided into two cases, i.e. when $C(t)\neq 0$ and when $C(t)=0$. Recall that $\{t_n\}_{n\in\mathbb{Z}}$ denotes the set of all eigenvalues of Π_r.

Theorem 3.1. *Assume that $C(t)\neq 0$, for all $t\in\mathbb{C}$. Let $g(\cdot)\in L^2(-\pi,\pi)$ and*

$$f(t)=\int_{-\pi}^{\pi} g(x)\phi(x,t)\,dx=\int_{-\pi}^{\pi} g(x)\left(e^{-itx}+\frac{\langle\varphi,r\rangle}{C(t)}P(x,t)\right)dx,\quad t\in\mathbb{C}, \tag{3.1}$$

where φ, P and Δ are the functions given in (2.10), (2.4) *and* (2.13) *respectively. Then $f(t)$ is an entire function of t that can be reconstructed via the sampling series*

$$f(t)=\sum_{n\in\mathbb{Z}} f(t_n)\frac{\Delta(t)}{(t-t_n)\Delta'(t_n)}. \tag{3.2}$$

The sampling series (3.2) *converges absolutely on $\mathbb{C}$ and uniformly on $\mathbb{R}$ and compact sets of the complex plane.*

Proof. Using integration by parts, one can derive the following Lagrange's identity

$$\langle\ell_r y,z\rangle=i\big[y(\cdot)\bar{z}(\cdot)\big]_{-\pi}^{\pi}+\langle y,\ell_r z\rangle, \tag{3.3}$$

for differentiable functions y, z. Let t and s be distinct complex numbers. Applying (3.3) with $y=\phi(\cdot,t)$ and $z=\phi(\cdot,s)$, we obtain

$$(t-s)\int_{-\pi}^{\pi}\phi(x,t)\bar{\phi}(x,s)\,dx=i\big(\phi(\pi,t)\bar{\phi}(\pi,s)-\phi(-\pi,t)\bar{\phi}(-\pi,s)\big). \tag{3.4}$$

Replacing s by t_n for some n, and using $\phi(\pi,t_n)=\phi(-\pi,t_n)$, then for, $t\neq t_n$,

$$\int_{-\pi}^{\pi}\phi(x,t)\phi(x,t_n)\,dx=i\bar{\phi}(\pi,t_n)\frac{\phi(\pi,t)-\phi(-\pi,t)}{t-t_n}=i\bar{\phi}(\pi,t_n)\frac{\Delta(t)}{t-t_n}. \tag{3.5}$$

Taking the limit in (3.5) as t approaches t_n, we have

$$\big\|\phi(\cdot,t_n)\big\|^2:=\int_{-\pi}^{\pi}\big|\phi(x,t)\big|^2\,dx=i\bar{\phi}(\pi,t_n)\Delta'(t_n). \tag{3.6}$$

Obviously

$$\phi(-\pi,t_n)=\varphi(-\pi,t_n)=e^{i\pi t_n}\neq 0,\quad n\in\mathbb{Z}.$$

Thus $\phi(\pi,t_n)=\phi(-\pi,t_n)\neq 0$. Therefore, Eq. (3.6) shows that the eigenvalues are all simple zeros of $\Delta(t)$ (algebraically), since the left-hand side also does not vanish because it is the norm of an eigenfunction. Since $\{\phi(\cdot,t_n)\}_{n\in\mathbb{Z}}$ is a complete orthogonal set of $L^2(-\pi,\pi)$, then applying Parseval's relation to (3.1) leads to

$$f(t)=\sum_{n\in\mathbb{Z}}\big\langle g(\cdot),\phi(\cdot,t_n)\big\rangle\frac{\langle\phi(\cdot,t),\phi(\cdot,t_n)\rangle}{\|\phi(\cdot,t_n)\|^2}=\sum_{n\in\mathbb{Z}} f(t_n)\frac{\langle\phi(\cdot,t),\phi(\cdot,t_n)\rangle}{\|\phi(\cdot,t_n)\|^2}. \tag{3.7}$$

Combining (3.5)–(3.7) we obtain (3.2) with a pointwise convergence on $\mathbb{C}$. It remains to prove the uniform and absolute convergence of (3.2). Let $\widetilde{\phi}_k(\cdot)=\phi(\cdot,t_k)/\|\phi(\cdot,t_k)\|$ and N be a positive integer. Using (3.7) and the Cauchy–Schwarz inequality to obtain

$$\left|f(t)-\sum_{|n|\leqslant N} f(t_n)\frac{\Delta(t)}{(t-t_n)\Delta'(t_n)}\right| = \left|f(t)-\sum_{|n|\leqslant N}\langle g,\widetilde{\phi}_n\rangle\langle\phi,\widetilde{\phi}_n\rangle\right|$$
$$\leqslant\left[\sum_{|n|>N}^{\infty}\left|\langle g,\widetilde{\phi}_n\rangle\right|^2\right]^{\frac{1}{2}}\left[\sum_{|n|>N}^{\infty}\left|\langle\phi,\widetilde{\phi}_n\rangle\right|^2\right]^{\frac{1}{2}}, \quad t\in\mathbb{C}. \tag{3.8}$$

As for uniform convergence on $\mathbb{R}$, in view of Bessel's inequality we have

$$\sum_{|n|>N}^{\infty}\left|\langle g,\widetilde{\phi}_k\rangle\right|^2\to 0 \quad \text{as } N\to\infty.$$

Moreover, from Bessel's inequality, we have

$$\sum_{|n|>N}^{\infty}\left|\langle\phi,\widetilde{\phi}_n\rangle\right|^2\leqslant\left\|\phi(\cdot,t)\right\|^2.$$

To prove uniform convergence on $\mathbb{R}$, it suffices to prove that $\|\phi(\cdot,t)\|$ is bounded on $\mathbb{R}$. Indeed, for $t\in\mathbb{R}$, we have

$$\left\|\phi(\cdot,t)\right\|^2=\int_{-\pi}^{\pi}\left|\phi(x,t)\right|^2dx=\int_{-\pi}^{\pi}\left|\left(\varphi(x,t)+\frac{\langle\varphi,r\rangle}{C(t)}P(x,t)\right)\right|^2dx.$$

Since $\varphi(x,t)=e^{-ixt}$ is bounded on $\mathbb{R}$ and

$$\left|\langle\varphi,r\rangle\right|=\left|\int_{-\pi}^{\pi}e^{-ixt}r(x)\,dx\right|\leqslant\left\|r(\cdot)\right\|_{L^1(-\pi,\pi)}, \tag{3.9}$$

$$\left|P(x,t)\right|=\left|ie^{-ixt}\int_{-\pi}^{x}e^{-i\tau t}r(\tau)\,d\tau\right|\leqslant\left\|r(\cdot)\right\|_{L^1(-\pi,\pi)}, \tag{3.10}$$

then our task will be achieved if we prove that $|1/C(t)|$ is bounded on $\mathbb{R}$. Indeed, since $C(t)\neq 0$ on $\mathbb{R}$; $C(t)$ is continuous on $\mathbb{R}$ and $\lim_{t\to\pm\infty}C(t)=1$, then we can find $L>0$ for which $|C(t)|\geqslant 1/2$, $|t|>L$. We claim that there exists $\eta>0$, such that $|C(t)|\geqslant\eta$, $|t|\leqslant L$ since otherwise we can find a sequence $\{s_n\}\subset[-L,L]$ for which $\lim_{n\to\infty}C(s_n)=0$. Since $\{s_n\}$ is bounded, then by Bolzano–Weierstrass' Theorem $\{s_n\}$ contains a convergent subsequence $\{s_{n_k}\}$ for which $\lim_{k\to\infty}s_{n_k}=s_0\in[-L,L]$. By continuity, $C(s_0)=0$, contradicting the assumption. Thus

$$\left|C(t)\right|\geqslant\max_{t\in\mathbb{R}}\{1/2,\eta\},\quad t\in\mathbb{R},$$

proving the uniform convergence of (3.2) on $\mathbb{R}$. In a similar manner, inform convergence can be proved on compact subsets of $\mathbb{C}$. Uniform convergence of (3.2) on compact subsets of $\mathbb{C}$ implies that f is entire of t. The absolute convergence on $\mathbb{R}$ holds since for $t\in\mathbb{C}$, Parseval's identity implies that

$$\sum_{n=-\infty}^{\infty}\left|f(t_n)\frac{\Delta(t)}{(t-t_n)\Delta'(t_n)}\right|=\left|\sum_{n=-\infty}^{\infty}\left|\langle g,\widetilde{\phi}_n\rangle\langle\phi,\widetilde{\phi}_n\rangle\right|\right|$$
$$\leqslant\left[\sum_{n=-\infty}^{\infty}\left|\langle g,\widetilde{\phi}_n\rangle\right|^2\right]^{\frac{1}{2}}\left[\sum_{n=-\infty}^{\infty}\left|\langle\phi,\widetilde{\phi}_n\rangle\right|^2\right]^{\frac{1}{2}}<\infty. \qquad \square \tag{3.11}$$

Now we study the sampling problem associated with Π_r when $C(t)=0$. In this case we may have double eigenvalues. Therefore, the sampling problem may be treated either in the view of [4,16,17], or by the use of Green's function [6]. In the following we use the last technique. We introduce a sampling theorem associated with Green's function of problem Π_r above. Since an eigenvalue t_n may have more than one linearly independent eigenfunctions, then expansion (2.36) has the form, $t\in\mathbb{C}$,

$$H(x,\xi,t)=\sum_{n\in\mathbb{Z}}\sum_{\nu=1}^{\nu_n}\frac{\phi_{n,\nu}(x)\overline{\phi}_{n,\nu}(\xi)}{t_n-t},\quad t\neq t_n, \tag{3.12}$$

where ν_n is the multiplicity of t_n and $\{\phi_{n,\nu}(x)\}_{\nu=1}^{\nu_n}$ is now a normalized set of eigenfunctions corresponding to t_n. From the theory developed above $1 \leqslant \nu_n \leqslant 2$. Let $\xi_0 \in [-\pi,\pi]$ be such that $\phi_n(\xi_0) \neq 0$ for all n. Such an ξ_0 exists since an eigenfunction may vanish only on a subset of measure zero of $[-\pi,\pi]$. Define the function $H_0(x,t)$ to be

$$H_0(x,t) := H(x,\xi_0,t), \quad t \in \mathbb{C} - \{t_n\}_{n\in\mathbb{Z}}. \tag{3.13}$$

Since the set of eigenfunctions is a complete orthonormal set of $L^2(-\pi,\pi)$, (3.12) can be viewed as the Fourier expansion of $H_0(x,t)$ with the Fourier coefficients

$$\frac{\overline{\phi}_{n,\nu}(\xi_0)}{t_n - t}, \quad t \neq t_n.$$

Also, $H_0(x,t)$ is a meromorphic function with simple poles t_n. The residue at each pole t_n is

$$r_n = \sum_{\nu=1}^{\nu_n} \phi_{n,\nu}(x)\overline{\phi}_{n,\nu}(\xi_0). \tag{3.14}$$

We start our analysis with the case when $C(t)$ and $\Delta(t)$ do not have common zeros. Define the function $\omega(t)$, $t \in \mathbb{C}$ to be

$$\omega(t) = \begin{cases} \Delta(t)C(t), & \text{if the zeros of } C(t) \text{ are different from those of } \Delta(t), \\ \Delta(t), & \text{if all zeros of } C(t) \text{ are also zeros of } \Delta(t). \end{cases} \tag{3.15}$$

Lemma 3.2. *The eigenvalues of problem Π_r are simple zeros of $\omega(t)$.*

Proof. From the proof of the previous theorem, all zeros of $\Delta(t)$ which are not zeros of $C(t)$ are simple. Now we prove that all zeros of $C(t)$ are algebraically simple. Indeed let t^* be a real zero of $C(t)$. From the results of the previous section $P(\cdot,t^*)$ is an eigenfunction of Π_r corresponding to t^*. From Lagrange's identity (3.3), we have for $t \in \mathbb{R}$, $t \neq t^*$,

$$\langle \ell_r(P(\cdot,t^*)), P(\cdot,t^*)\rangle = i\big[P(\cdot,t)\overline{P}(\cdot,t^*)\big]_{-\pi}^{\pi} + \langle P(\cdot,t), \ell_r(P(\cdot,t^*))\rangle.$$

Since, in this case $P(-\pi,t^*) = P(\pi,t^*) = 0$ and $P(\cdot,t)$ satisfies (2.5), then

$$\begin{aligned} \ell_r(P(\cdot,t)) &= iP'(x,t) + r(x)\int_{-\pi}^{\pi} r(\tau)P(\tau,t)\,d\tau \\ &= -r(x) + tP(x,t) + r(x)\langle P(\cdot,t), r(\cdot)\rangle. \end{aligned}$$

Hence

$$\begin{aligned} (t-t^*)\langle P(\cdot,t), P(\cdot,t^*)\rangle &= \langle (-r(\cdot) + r(\cdot)\langle P(\cdot,t), r(\cdot)\rangle), P(\cdot,t^*)\rangle \\ &= -\langle r(\cdot), P(\cdot,t^*)\rangle\{1 - \langle P(\cdot,t), r(\cdot)\rangle\} \\ &= -1 + \langle P(\cdot,t), r(\cdot)\rangle = -C(t), \end{aligned}$$

since $\langle P(\cdot,t^*), r(\cdot)\rangle = 1 = \langle r(\cdot), P(\cdot,t^*)\rangle$. Letting $t \to t^*$ in the last equation, yields

$$\|P(\cdot,t^*)\|^2 = -C'(t^*).$$

Since $P(\cdot,t^*)$ is an eigenfunction, then $C'(t^*) \neq 0$. Hence all zeros of $C(t)$ are simple and the proof is complete. □

The previous lemma indicated an interesting phenomenon, namely the fact that all zeros of $C(t)$ are algebraically simple although some of them might be geometrically double. In fact it is not hard to see that a zero of $C(t)$ is geometrically double if and only if it is also a zero of $\Delta(t)$. Example 4.2 below exhibits this phenomenon. Using the same technique we can see that the real zeros of $C(t)$ are simple zeros of $V(P(\cdot,t))$. From (2.12), we have

$$V(\phi(\cdot,t)) = V(\varphi(\cdot,t)) + \frac{\langle \varphi(\cdot,t), r(\cdot)\rangle}{C(t)} V(P(\cdot,t)).$$

Hence $\Delta(t)$ is analytic at the real zeros of $C(t)$. Now define the entire function

$$\Phi(x,t) := \omega(t)H_0(x,t), \quad t \in \mathbb{C}. \tag{3.16}$$

The second sampling theorem of this paper is the following. It gives another perturbed WKS sampling theorem, which is a perturbed version of that derived by Haddad et al. [19].

Theorem 3.3. *Let $g \in L^2(-\pi,\pi)$ and*

$$F(t) = \int_{-\pi}^{\pi} \bar{g}(x)\Phi(x,t)\,dx, \quad t \in \mathbb{R}. \tag{3.17}$$

Then $F(t)$ is an entire function that admits the sampling representation

$$F(t) = \sum_{n\in\mathbb{Z}} F(t_n)\frac{\omega(t)}{(t-t_n)\omega'(t_n)}. \tag{3.18}$$

The sampling series (3.18) *is absolutely and uniformly convergent on compact subsets of $\mathbb{C}$ and uniformly on $\mathbb{R}$.*

Proof. Since both g and Φ are L^2-functions and $\{\phi_n(\cdot)\}_{n\in\mathbb{Z}}$ is a complete orthonormal set in $L^2(-\pi,\pi)$, then

$$g(x) = \sum_{n\in\mathbb{Z}} \langle g, \phi_n\rangle \phi_n(x), \qquad \Phi(x,t) = \sum_{n\in\mathbb{Z}} \langle \Phi, \phi_n\rangle \phi_n(x) \tag{3.19}$$

are the Fourier series of g and Φ, respectively. Here $\langle g, \phi_n\rangle$ and $\langle \Phi, \phi_n\rangle$ are the Fourier coefficients. Using Parseval's identity, we get

$$F(t) = \sum_{n\in\mathbb{Z}} \overline{\langle g, \phi_n\rangle}\langle \Phi, \phi_n\rangle. \tag{3.20}$$

In the view of (3.12) above, Eq. (3.20) can be rewritten in the form

$$F(t) = \sum_{n\in\mathbb{Z}}\sum_{\nu=1}^{\nu_n} \overline{\langle g, \phi_{n,\nu}\rangle}\langle \Phi, \phi_{n,\nu}\rangle. \tag{3.21}$$

From the definition of Φ, we obtain for $t \in \mathbb{C}$, $t \neq t_n$,

$$\langle \Phi, \phi_{n,\nu}\rangle = \frac{\omega(t)}{t_n - t}\bar{\phi}_{n,\nu}(\xi_0). \tag{3.22}$$

Since

$$F(t) = \omega(t)\int_{-\pi}^{\pi} \bar{g}(x)H_0(x,t)\,dx, \tag{3.23}$$

and $H_0(x,t)$ has simple poles at the eigenvalues with the residues (3.12), then

$$\begin{aligned} F(t_n) &= \lim_{t\to t_n}\frac{\omega(t)}{t-t_n}\int_{-\pi}^{\pi}(t-t_n)\bar{g}(x)H_0(x,t)\,dx \\ &= -\omega'(t_n)\sum_{n=1}^{\nu_n}\bar{\phi}_{n,\nu}(\xi_0)\int_{-\pi}^{\pi}\bar{g}(x)\phi_{n,\nu}(x)\,dx \\ &= -\omega'(t_n)\sum_{n=1}^{\nu_n}\bar{\phi}_{n,\nu}(\xi_0)\overline{\langle g, \phi_{n,\nu}\rangle}. \end{aligned} \tag{3.24}$$

Substituting from (3.22) and (3.24) in (3.21), one gets (3.18).

The proof of uniform and absolute convergence on $\mathbb{R}$ can be established with a slight modification of that of the previous theorem. As for uniform and absolute convergence on compact subsets of $\mathbb{C}$, we use the same arguments of the proof of the previous theorem and the identity of [13, p. 50] that guarantees the boundedness of $\|\Phi(\cdot,t)\|$ on compact subsets of $\mathbb{C}$. □

Remarks.

- As in the classical result in sampling theory, we could define the kernel in terms of a canonical product. The problem in this setting is that we do not know the exact asymptotic behavior of the eigenvalues. We conjecture that

$$t_n \sim n \quad \text{as } |n| \to \infty.$$

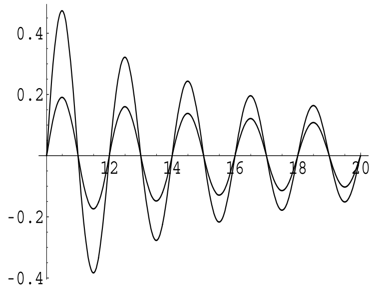

Fig. 1. $\Re f_p(t)$ and $\Re f_u(t)$, $10 < t < 20$.

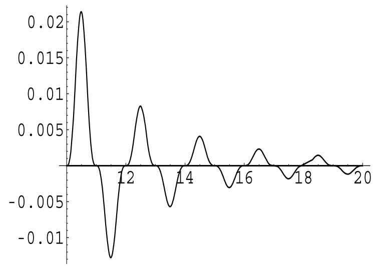

Fig. 2. $\Im f_p(t)$ and $\Im f_u(t)$, $10 < t < 20$.

However, from the theory of Fredholm integral operators [13], we know at least that

$$\sum_{n=-\infty}^{\infty} \frac{1}{|t_n|^2} < \infty,$$

provided that zero is not among $\{t_n\}_{n\in\mathbb{Z}}$. Therefore we can define the entire function

$$\Phi(x,t) = \left[t \prod_{n=-\infty,\, n\neq 0}^{\infty} \left(1 - \frac{t}{t_n}\right) \exp(t/t_n) \right] H_0(x,t), \quad t \in \mathbb{C},$$

if $t_0 = 0$ is an eigenvalue, and

$$\Phi(x,t) = \left[\prod_{n=-\infty}^{\infty} \left(1 - \frac{t}{t_n}\right) \exp(t/t_n) \right] H_0(x,t), \quad t \in \mathbb{C},$$

if zero is not an eigenvalue. Then we can derive another sampling theorem for integral transforms of the type (3.17).

- The case when $C(t)$ and $\Delta(t)$ have common zeros will be treated as before. We only modify the definition of the function $\omega(t)$ as follows. Let us denote the zeros of $C(t)$ which are not zeros of $\Delta(t)$ by $s_1, \ldots, s_m$. Then define $\omega(t)$ to be

$$\omega(t) = C(t)\Delta(t) \left[\prod_{l=1}^{m} \left(1 - \frac{t}{s_l}\right) \right]^{-1}.$$

Then the analysis will be completed similarly.

- The analysis considered above could be extended in many directions, leading to sampling results for more general integral transforms. For example, the integro-differential equation (2.1) could be extended so that $r(\cdot)$ is not necessarily real valued. Also the boundary condition (1.4) could be extended without affecting the self-adjointness. Thus problem (2.1), (1.4) could be extended to

$$\ell_r(y) := iy' + q(x)y + r(x) \int_{-\pi}^{\pi} \bar{r}(\tau) y(\tau)\, d\tau = ty, \quad -\pi \leqslant x \leqslant \pi, \tag{3.25}$$

$$V(y) := y(\pi) - e^{i\theta} y(-\pi) = 0. \tag{3.26}$$

Here $r(\cdot)$ is a complex-valued continuous function; $q(\cdot)$ is real valued continuous functions and θ is a real number.

- The integral transform of the first sampling theorem is a family of transforms based on the different choices of $r(\cdot)$. The kernel $\phi(\cdot, t)$ is nothing but,

$$\phi(x,t) = e^{-ixt} \left[1 + i \frac{\langle e^{-ixt}, r(x)\rangle}{1 - \langle P(\cdot,t), r(\cdot)\rangle} \int_{-\pi}^{x} r(\tau) e^{i\tau t}\, d\tau \right]. \tag{3.27}$$

4. Examples and comparisons

In the following we give two concrete examples illustrating the results of the previous sections. In the first example $C(t) \neq 0$, $t \in \mathbb{C}$, and in the second one $C(t) = 0$ for some $t \in \mathbb{R}$. In both examples all eigenvalues can be computed explicitly, like the unperturbed case. We illustrate the differences between the perturbed and the unperturbed transforms of the first example via Figs. 1 and 2.

Example 4.1. Consider the following problem

$$iy' + \int_{-\pi}^{\pi} y(\tau)\,d\tau = ty, \qquad y(\pi) - y(-\pi) = 0. \tag{4.1}$$

According to the previous notations,

$$P(x,t) = \frac{1 - e^{-it(x+\pi)}}{t}, \qquad C(t) = 1 + \frac{i}{t^2}\big(2it\pi + e^{-2it\pi} - 1\big), \tag{4.2}$$

$$\phi(x,t) = e^{-itx} + \frac{2\sin t\pi\,(1 - e^{-it(x+\pi)})}{t^2 C(t)}, \qquad \Delta(t) = -2i\sin t\pi\left(1 + i\frac{1 - e^{-2it\pi}}{t^2 C(t)}\right). \tag{4.3}$$

One can check that the eigenvalues are $t_n = n$, $n \in \mathbb{Z}^* := \mathbb{Z} - \{0\}$, $t_0 = 2\pi$, and $C(t)$ has no real zeros. We also have

$$\Delta'(t) = \begin{cases} -2i\pi(-1)^n, & t = n \in \mathbb{Z}^*, \\ \frac{-4\pi\sin 2\pi^2}{e^{-4i\pi^2} - 1}, & t = 2\pi. \end{cases} \tag{4.4}$$

Following Theorem 3.1 above, the transform

$$f(t) = \int_{-\pi}^{\pi} g(x)\left(e^{-itx} + \frac{2\sin t\pi\,(1 - e^{-it(x+\pi)})}{t^2 - 2\pi t + i(e^{-2i\pi t} - 1)}\right)dx, \quad t \in \mathbb{R}, \tag{4.5}$$

has the sampling form

$$f(t) = f(2\pi)\left(1 + \frac{i(1 - e^{-2it\pi})}{t^2 C(t)}\right)\frac{i\sin t\pi\,(e^{-4i\pi^2} - 1)}{2\pi\,\sin 2\pi^2(t - 2\pi)} + \sum_{n\in\mathbb{Z}^*} f(n)\left(1 + \frac{i(1 - e^{-2it\pi})}{t^2 C(t)}\right)\frac{\sin\pi(t-n)}{\pi(t-n)}. \tag{4.6}$$

Now we illustrate the figures of the perturbed transform (4.5) and the unperturbed one when $g(x) \equiv 1$. Let f_u and f_p denote the unperturbed and perturbed transforms respectively. We have

$$f_u(t) = \int_{-\pi}^{\pi} e^{-itx}\,dx = \frac{2\sin t\pi}{t}, \tag{4.7}$$

$$\begin{aligned} f_p(t) &= \int_{-\pi}^{\pi}\left(e^{-itx} + \frac{2\sin t\pi\,(1 - e^{-it(x+\pi)})}{t^2 - 2\pi t + i(e^{-2i\pi t} - 1)}\right)dx \\ &= \frac{2t\sin t\pi\,[(t^2 - 2t\pi + \sin 2t\pi) + i(1 - \cos 2t\pi)]}{(1 - \cos 2t\pi)^2 + (t^2 - 2t\pi + \sin 2t\pi)^2}. \end{aligned} \tag{4.8}$$

Figs. 1 and 2 illustrate $\Re f_p(t)$, $\Re f_u(t)$ and $\Im f_p(t)$, $\Im f_u(t)$ respectively. Notice that $\Im f_u(t) \equiv 0$ on $\mathbb{R}$ and $\Im f_p(t)$ is very small. Also $\Re f_u(t)$ and $\Re f_p(t)$ eventually merge with each other. This can be justified via the following asymptotic analysis. Letting $f_\varepsilon(t) := f_p(t) - f_u(t)$, then

$$\begin{aligned} \big|f_\varepsilon(t)\big| &= \left|\int_{-\pi}^{\pi} \frac{2\sin t\pi\,(1 - e^{-it(x+\pi)})}{t^2 - 2\pi t + i(e^{-2i\pi t} - 1)}\,dx\right| \\ &\leqslant \frac{8\pi}{|t^2 - 2\pi t + i(e^{-2i\pi t} - 1)|}. \end{aligned}$$

Therefore

$$f_p(t) = f_u(t) + \mathcal{O}\big(t^{-2}\big), \quad \text{as } |t| \to \infty.$$

Example 4.2. Here we consider a problem where $C(t) = 0$, which is

$$iy' + \frac{1}{2\pi}\int_{-\pi}^{\pi} y(\tau)\,d\tau = ty, \qquad y(\pi) - y(-\pi) = 0. \tag{4.9}$$

In this case we have,

$$P(x,t) = \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-it(x+\pi)}}{t}\right), \qquad C(t) = 1 + \frac{i}{2\pi t^2}\left(2it\pi + e^{-2it\pi} - 1\right), \tag{4.10}$$

$$\phi(x,t) = e^{-itx} + \frac{\sin t\pi\,(1-e^{-it(x+\pi)})}{\pi t^2 C(t)}, \qquad \Delta(t) = -2i\sin t\pi\left(1 + \frac{i(1-e^{-2it\pi})}{2\pi t^2 C(t)}\right). \tag{4.11}$$

One sees that the eigenvalues are the zeros of $\Delta(t)$, which are $t_n = n$, $n \in \mathbb{Z}^*$ with the corresponding eigenfunctions $\{e^{-inx}\}_{n\in\mathbb{Z}^*}$ and $t_0 = 1$, the only real zero of $C(t)$, with corresponding eigenfunctions $\phi(x,1) = e^{-ix}$ and $P(x,1) = \frac{1-e^{-i(x+\pi)}}{\sqrt{2\pi}}$. This means that $t_0 = 1$ is a double eigenvalue of the problem. Also $\omega(t) = \Delta(t)$, and

$$\Delta'(t_n) = \begin{cases} -2i\pi(-1)^n, & n \in \mathbb{Z}^*,\ n \neq 1, \\ i\pi, & n = 1. \end{cases} \tag{4.12}$$

For $\xi_0 \in [-\pi,\pi]$, a transform defined as in Theorem 3.3 above has the sampling representation

$$F(t) = F(1)\left(1 + \frac{i(1-e^{-2it\pi})}{2\pi t^2 C(t)}\right)\frac{-2\sin\pi(t-1)}{\pi(t-1)} + \sum_{n\in\mathbb{Z}^*} F(n)\left(1 + \frac{i(1-e^{-2it\pi})}{2\pi t^2 C(t)}\right)\frac{\sin\pi(t-n)}{\pi(t-n)}, \quad t \in \mathbb{C}. \tag{4.13}$$

Notice that in the previous example the condition

$$\langle\varphi(\cdot,1), r(\cdot)\rangle = \int_{-\pi}^{\pi} \exp(-ix)\,dx = 0$$

is fulfilled.

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